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Generalized Analytic Continuation

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ABSTRACT

Analytic continuation is the extension of the domain of a given analytic function in the complex plane, to a larger domain of the complex plane. This process has been utilized in many other areas of mathematics, and has given mathematicians new insight into some of the world's hardest problems. This paper will cover more general forms of analytic continuation, which will be referred to as *generalized analytic continuations*. The paper will closely follow William Ross' and Harold Shapiro's book "Generalized Analytic Continuation" [14], with the proofs worked out with more detail, and a few generalizations are made regarding the Poincaré example in Section 3.3.

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1. INTRODUCTION

Analytic continuation is a technique to extend the domain of a given analytic function in the complex plane. It has appeared in many other areas of mathematics as well, appearing in operator theory, differential equations, and even the Riemann Zeta function. This paper will provide an introduction to the classical analytic continuation, then go into two other, more generalized forms of analytic continuations. Thus, they are called generalized analytic continuations. The first generalized analytic continuation that will be covered is formed by matching non-tangential limits on the *natural boundaries* of an analytic continuation, which was first presented by Poincaré in 1883. We then bridge into *almost periodic* sequences and functions, and touch upon their properties which will give a form of generalized analytic continuation also. The last section of this paper gives several conditions for which no form of an analytic continuation exists for the generalized analytic continuations discussed in the first three sections of this paper.

Definition 1. Suppose f is an analytic function defined on a connected open subset $U \subseteq \mathbb{C}$. Let $U \subset V$, where V is an open connected subset of \mathbb{C} , and F is an analytic function defined on V such that,

$$F(z) = f(z), \quad \forall z \in U,$$

then F is called an analytic continuation of f to V . By the identity theorem, F is unique.

This is the idea of classical analytic continuation to a larger domain in \mathbb{C} . The boundary ∂U of the function's domain is called the *natural boundary* if $f|_U$ does not have an analytic continuation to any larger domain. Analytic continuation appears in various areas of mathematics, one of which is the Riemann Zeta function. We will now look at a few classical examples of analytic continuation.

We introduce the following notation for the open unit disk,

$$\mathbb{D} = \{z : |z| < 1\},$$

for the unit circle,

$$\mathbb{T} = \{z : |z| = 1\},$$

and for the open disk of radius b , centered at a ,

$$D(a, b) = \{z : |z - a| < b\}.$$

Example 1.

$$f(z) = \sum_{n=0}^{\infty} z^n, \quad z \in \mathbb{D}.$$

This series converges everywhere in \mathbb{D} , but diverges on the $\mathbb{C} \setminus \mathbb{D}$. The series converges to

$$\frac{1}{1 - z},$$

and this function is defined everywhere except the point $z = 1$, so this is the classical analytic continuation of f across $\mathbb{T} \setminus \{1\}$ to $\mathbb{C} \setminus \{1\}$.

Example 2. *The Schwarz Reflection Principle [15, Theorem 11.14 (p. 237)]*

Suppose L is a segment of the real axis, Ω^+ is a domain, that is, an open connected subset of \mathbb{C} in the half-plane $\Pi^+ = \{z : \operatorname{Im} z > 0\}$, and every $t \in L$ is the center of an open disk D_t such that $\Pi^+ \cap D_t$ lies in Ω^+ . Let Ω^- be the reflection of Ω^+ across \mathbb{R} :

$$\Omega^- = \{z : \bar{z} \in \Omega^+\}.$$

Suppose $f = u + iv$ is holomorphic in Ω^+ , and

$$\lim_{n \rightarrow \infty} v(z_n) = 0$$

for every sequence $\{z_n\}$ in Ω^+ which converges to a point of L . Then there is a function F , holomorphic in $\Omega^+ \cup L \cup \Omega^-$, such that $F(z) = f(z)$ in Ω^+ ; this F satisfies the relation

$$F(\bar{z}) = \overline{F(z)} \quad (z \in \Omega^+ \cup L \cup \Omega^-).$$

This theorem says that a holomorphic function that has real limits on the real axis can be extended to a holomorphic function on the symmetric region about the real axis.

The last example that will be presented is the analytic continuation of the Γ function.

Example 3.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \operatorname{Re} z > 0.$$

Let

$$\Gamma(z) = P(z) + Q(z),$$

where

$$P(z) = \int_0^1 e^{-t} t^{z-1} dt \text{ and } Q(z) = \int_1^\infty e^{-t} t^{z-1} dt.$$

By replacing e^{-t} by its Taylor expansion in the integral representation of $P(z)$ and integrating term-by-term, the following expression is obtained,

$$P(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(z+k)k!}.$$

Note that this expression is valid for $\operatorname{Re} z > 0$ and converges uniformly and absolutely on compact subsets of $\mathbb{C} \setminus \{0, -1, -2, \dots\}$. Also, $P(z)$ has poles at $z = 0, -1, -2, \dots$. Thus, $P(z)$ represents a meromorphic function, and the analytic continuation of $\Gamma(z)$ is given by

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(z+k)k!} + \int_1^\infty e^{-t} t^{z-1} dt.$$

Thus, it is possible to find a meromorphic continuation of $\Gamma(z)$ to the entire complex plane, with poles at the negative integers and at the origin.

In the sequel, we will consider Chapters 3, 7, and parts of Chapter 6 from William T. Ross and Harold S. Shapiro's work ([14]), together with some background information and theorems from other sources including W. Rudin [15], and C. Corduneanu ([4]). In the first two sections we will begin by taking an in-depth look at two types of "generalized classical analytic continuations." The first section

will look at a result from Poincaré, followed by the same idea with almost periodic functions in the second section. After that, we will do an overview of another type of generalized analytic continuation called pseudocontinuation with gap series (see Section 5).

Generalized analytic continuation (GAC) investigates the relationship of the component functions on the interior and exterior of a closed curve, in certain cases where the classical notion of analytic continuation says there is a natural boundary. The following is an excerpt from É. Borel's work [[2], p. 100], where he began to study some of these ideas.

“...we wished only to show how one could introduce into the calculations analytic expressions whose values, in different regions of their domain of convergence, are mutually linked in a simple way. It seems, on the basis of that, that one could envision extending Weierstrass' definition of analytic function and regarding in certain cases as being [parts of] the same function, analytic functions having separate domains of existence. But for that it is necessary to impose restrictions on the analytic expressions one considers, and because he did not wish to impose such restrictions Weierstrass answered in the negative [this] question:

“Therefore the thought was not to be ignored, as to whether in the case where an arithmetic expression $F(x)$ represents different monogenic functions in different portions of its domain of validity, there is an essential connection, with the consequence that the properties of the one should determine the properties of the other. Were this the case, it would follow that the concept ‘monogenic function’ must be widened.”-(Weierstrass, *Mathematische Werke*, col. 2, p. 212)

It is not possible for us to give to this Chapter a decisive conclusion; for, in our opinion, the question addressed here is not entirely resolved and calls for further research. We would be content if we have convinced our readers that neither the fundamental works of Weierstrass, nor the later ones of Mittag-Leffler, Appell, Poincaré,

Runge, Painlevé entirely answer the question as to the relations between the notions of analytic function, and analytic expression. One can even say without exaggeration, that the classification of analytic expressions which are incapable of representing zero [on some domain] without doing so everywhere, is yet to be brought to completion.”

The idea of a coherence property is a natural relationship of a function and its extension by some means of matching the functions along some shared boundary. This was investigated in the 1920’s and 1930’s by Walsh ([18]) and more recently in the 1960’s and 1970’s by Tumarkin ([17]) and Gončar ([9]). Tumarkin’s and Walsh’s explorations of the geometric restriction of the location of poles, without regard to the rate of convergence, have recently been shown to have a surprising connection with a problem in operator theory; namely, the classification of the cyclic vectors for the backward shift operator on the Hardy spaces. This was discovered by Ross and his co-workers ([14]). There has also been a use of GAC in the research of the backward shift in other function spaces such as the Bergman and Dirichlet spaces. The employment of GAC in other areas is also present. One such case is in the research on the study of electrical networks (the Darlington synthesis problem which is briefly introduced in Chapter 6, §6.7 [14]) and is related to linear differential equations of infinite order (Chapter 6, §6.8 [14]).

While the idea of GAC is not completely understood yet, progress is being made. There are questions still to be answered and explored, such as the different types of coherence and different types of strategy with GAC. The goal of some mathematicians is to fully understand the concepts and limitations of GAC as well as the concepts of divergent series. Ross and Shapiro expressed that their book [14] would offer a humble beginning to the understanding and development of GAC.

2. BACKGROUND INFORMATION

Before we discuss the subject of GAC, there are some preliminary concepts and theorems we will need.

Definition 2. A nontangential approach region with vertex 1 is denoted by Ω_α , where $0 < \alpha < 1$, and Ω_α is the union of the disk $D(0, \alpha)$ and the line segments from $z = 1$ to points of $D(0, \alpha)$. (This is also called a classical Stolz region.)

Definition 3. A function F , defined in \mathbb{D} , is said to have nontangential limit λ at $e^{i\theta} \in \mathbb{T}$ if for each $\alpha < 1$,

$$\lim_{j \rightarrow \infty} F(z_j) = \lambda$$

for every sequence $\{z_j\}$ that converges to $e^{i\theta}$ and that lies in $e^{i\theta}\Omega_\alpha$.

For any continuous function f in \mathbb{D} , we will use the following notation

$$f_r(e^{i\theta}) = f(re^{i\theta}), \quad (0 \leq r < 1).$$

Let σ denote the Lebesgue measure on \mathbb{T} normalized so $\sigma(\mathbb{T}) = 1$. Henceforth, L^p -norms will refer to $L^p(\sigma)$ norms; that is

$$\|f_r\|_p = \left(\int_{\mathbb{T}} |f_r|^p d\sigma \right)^{1/p} \quad (0 < p < \infty), \quad \|f_r\|_\infty = \sup_{\theta} |f(re^{i\theta})|.$$

Definition 4. If $f \in \mathfrak{H}(\mathbb{D})$ (holomorphic on \mathbb{D}) and $0 < p \leq \infty$, we put

$$\|f\|_p = \sup\{\|f_r\|_p : 0 \leq r \leq 1\}.$$

If $0 < p \leq \infty$, H^p is defined to be the class of all $f \in \mathfrak{H}(\mathbb{D})$, where $\|f\|_p < \infty$.

Theorem 1. (The Lebesgue-Radon-Nikodym Theorem [15, Theorem 6.10])

Let μ be a positive σ -finite measure on a σ -algebra \mathcal{M} in a set X , and let λ be a complex measure on \mathcal{M} . Then the following statements hold.

(a) There is a unique pair of complex measures λ_a and λ_s on \mathcal{M} such that

$$\lambda = \lambda_a + \lambda_s, \quad \lambda_a \ll \mu, \quad \lambda_s \perp \mu.$$

If λ is positive and finite, then so are λ_a and λ_s .

(b) There is a unique $h \in L^1(\mu)$ such that

$$\lambda_a(E) = \int_E h \, d\mu$$

for every set $E \in \mathcal{M}$.

Definition 5. Let μ be a bounded complex regular Borel measure on \mathbb{T} . The Cauchy transform of μ is the analytic function defined on $\mathbb{C}_\infty \setminus \mathbb{T}$ by

$$f_\mu(z) := \int_{-\pi}^{\pi} \frac{1}{1 - e^{-it}z} d\mu(e^{it}).$$

Theorem 2. ([15, Theorem 5.25])

Suppose \mathfrak{A} is a vector space of continuous complex-valued functions on \mathbb{D}^- . Suppose \mathfrak{A} contains all polynomials. If

$$\sup_{z \in \mathbb{D}} |f(z)| = \sup_{z \in \mathbb{T}} |f(z)|.$$

for every $f \in \mathfrak{A}$, then the Poisson integral representation

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} f(e^{it}) dt \quad (z = re^{i\theta})$$

is valid for every $f \in \mathfrak{A}$ and every $z \in \mathbb{D}$.

Furthermore, if μ is a complex Borel measure on \mathbb{T} , then

$$F(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} d\mu(e^{it}),$$

is called the Poisson integral of μ , and is denoted by $P[d\mu]$.

Theorem 3. (Fatou's Theorem [15, Theorem 11.24])

If $d\mu = f d\sigma + d\mu_s$ is the Lebesgue decomposition of a complex Borel measure μ on \mathbb{T} , where $f \in L^1(\mathbb{T})$ and $\mu_s \perp \sigma$, then $P[d\mu]$ has nontangential limit $f(e^{i\theta})$ at almost all $e^{i\theta} \in \mathbb{T}$.

Theorem 4. (Plessner's Theorem [3, 1927])

If $f \in \mathfrak{M}(\mathbb{D})$ (i.e., f is meromorphic on \mathbb{D}) has nontangential limits equal to zero on some set of positive measure in \mathbb{T} , then f must be identically zero on \mathbb{D} .

Theorem 5. (Lebesgue's Dominated Convergence Theorem [15, Theorem 1.34])

Let X be a measurable space. Suppose $\{f_n\}$ is a sequence of complex measurable functions on X such that

$$f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \leq g(x) \quad (n = 1, 2, 3, \dots; x \in X),$$

then $f \in L^1(\mu)$,

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| d\mu = 0,$$

and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

3. THE POINCARÉ EXAMPLE

3.1. Poincaré's Result. The following theorem was presented by Poincaré.

Theorem 6. *Let L be a smooth closed curve which bounds a convex set in the plane, $\{z_n\}$ be a sequence of dense distinct points on L , and $\{c_n\}$ be an absolutely summable sequence of non-zero complex numbers. Define f to be the function*

$$f(z) := \sum_{n=1}^{\infty} \frac{c_n}{z - z_n}, \quad z \notin L.$$

Then $f|_{\text{int}(L)}$ (f restricted to the interior of L) does not have an analytic continuation across any point of L .

Proof. Since $\{z_n\}$ is a sequence of distinct points in L , we can find a point w in the interior of L such that the closed disk centered at w with radius $|w - z_k| := R$ meets L only at z_k . Fix this w , and do a translation so that w is now at the origin. For $z \notin L$, let $\delta = \inf\{|z - z_k|\}$. Consider the series representation of f , and note that

$$\left| f(z) - \sum_{k=1}^n \frac{c_k}{z - z_k} \right| = \left| \sum_{k=n+1}^{\infty} \frac{c_k}{z - z_k} \right| \leq \sum_{k=n+1}^{\infty} \frac{|c_k|}{|z - z_k|} \leq \frac{1}{\delta} \sum_{k=n+1}^{\infty} |c_k|,$$

and this upper bound tends to 0 as $n \rightarrow \infty$, since $\{c_k\}$ is absolutely summable.

Thus, f converges uniformly for $z \notin L$ and by the *Weierstrass Convergence Theorem*, f is holomorphic. Since f is holomorphic in a neighborhood of the origin, it has a Taylor series representation about the origin. The first two Taylor coefficients $\{B_q\}$ are

$$\begin{aligned} B_0 = f(0) &= \frac{-c_1}{z_1} + \frac{-c_2}{z_2} + \frac{-c_3}{z_3} + \dots \\ B_1 = f'(0) &= \frac{-c_1}{z_1^2} + \frac{-c_2}{z_2^2} + \frac{-c_3}{z_3^2} + \dots \end{aligned}$$

and so on. In general,

$$B_q = \frac{f^{(q)}(0)}{q!} = - \sum_{n=0}^{\infty} \frac{c_n}{z_n^{q+1}}, \quad q = 0, 1, 2, \dots$$

Thus,

$$f(z) = \sum_{q=0}^{\infty} z^q B_q, \text{ and this series converges for } |z| < R.$$

We now show that this Taylor series has radius of convergence exactly R , and whence f cannot have an analytic continuation across z_k . If f has a radius of convergence greater than R , then the series will converge when $|z| = R$. So it will suffice to show that $|z^q B_q| = |R^q B_q|$ does not tend to 0 as $q \rightarrow \infty$, thus showing that the series does not converge. Note the following,

$$R^q B_q = - \frac{c_k R^q}{z_k^{q+1}} - \sum_{n \neq k} \frac{c_n R^q}{z_n^{q+1}}$$

For $n \neq k$, $\left| \frac{R}{z_n} \right| < 1$ since $z_n \notin B(0, R)^-$. Moreover, since $\{c_n\}$ is absolutely summable we obtain have the following:

$$\begin{aligned} \sum_{n \neq k} \left| \frac{c_n R^q}{z_n^{q+1}} \right| &< \sum_{n \neq k} \left| \frac{c_n R}{z_n^2} \right| \\ \Rightarrow R \sum_{n \neq k} \left| \frac{c_n R^q}{z_n^{q+1}} \right| &< \sum_{n \neq k} \left| \frac{c_n R^2}{z_n^2} \right| < \sum_{n \neq k} |c_n|. \end{aligned}$$

Looking at

$$g_q(n) := \left| \frac{c_n R^q}{z_n^{q+1}} \right|, \quad n, q \in \mathbb{N}, \quad n \neq k,$$

as a sequence of complex measurable functions on \mathbb{N} with counting measure μ ; we see that

$$\lim_{q \rightarrow \infty} \left| \frac{c_n R^q}{z_n^{q+1}} \right| = 0$$

and

$$g_q(n) = \left| \frac{c_n R^q}{z_n^{q+1}} \right| \leq \left| \frac{c_n}{z_n} \right| < \frac{1}{R} |c_n| \quad (q = 1, 2, 3, \dots; n \in \mathbb{N} \setminus k),$$

and

$$\sum_{n \neq k} \frac{1}{R} |c_n| = \frac{1}{R} \sum_{n \neq k} |c_n| < \infty.$$

Invoking *The Lebesgue Dominated Convergence Theorem* (cf. Theorem 5)

$$\lim_{q \rightarrow \infty} \sum_{n \neq k} g_q(n) = \sum_{n \neq k} \lim_{q \rightarrow \infty} g_q(n) = \sum_{n \neq k} \lim_{q \rightarrow \infty} \left| \frac{c_n R^q}{z_n^{q+1}} \right| = \sum_{n \neq k} 0 = 0$$

Therefore, since $|z_k| = R$,

$$\lim_{q \rightarrow \infty} |R^q B_q| = \lim_{q \rightarrow \infty} \left| -\frac{c_k R^q}{z_k^{q+1}} - \sum_{n \neq k} \frac{c_n R^q}{z_n^{q+1}} \right| = \lim_{q \rightarrow \infty} \left| \frac{-c_k R^q}{z_k^{q+1}} \right| \neq 0.$$

□

3.2. Matching nontangential boundary values.

Theorem 7. *Let $\{e^{i\theta_n}\}$ be a sequence of distinct points on \mathbb{T} and let $\{c_n\}$ be an absolutely summable sequence of complex numbers. The function*

$$f(z) := \sum_{n=1}^{\infty} \frac{c_n}{1 - ze^{-i\theta_n}}, \quad z \in \mathbb{D},$$

has the following property:

$$\lim_{r \rightarrow 1^-} (1-r)f(re^{i\theta_m}) = c_m, \quad m = 1, 2, 3, \dots$$

Proof. We have

$$\begin{aligned} \lim_{r \rightarrow 1^-} (1-r)f(re^{i\theta_m}) &= \lim_{r \rightarrow 1^-} (1-r) \sum_{n=1}^{\infty} \frac{c_n}{1 - re^{i\theta_m} e^{-i\theta_n}} \\ &= \lim_{r \rightarrow 1^-} \left[(1-r) \frac{c_m}{(1-r)} + (1-r) \sum_{n \neq m} \frac{c_n}{1 - re^{i\theta_m} e^{-i\theta_n}} \right]. \end{aligned}$$

Note that an upper estimation of $\lim_{r \rightarrow 1^-} \left[(1-r) \frac{c_m}{(1-r)} + (1-r) \sum_{n \neq m} \frac{c_n}{1 - re^{i\theta_m} e^{-i\theta_n}} \right]$ is

$$\frac{|1-r||c_n|}{|1 - re^{i\theta_m} e^{-i\theta_n}|} \leq \frac{|1-r||c_n|}{1 - |re^{i\theta_m} e^{-i\theta_n}|} = |c_n|.$$

So the *Lebesgue Dominated Convergence Theorem* yields

$$\lim_{r \rightarrow 1^-} (1-r) \sum_{n \neq m} \frac{c_n}{1 - re^{i\theta_m} e^{-i\theta_n}} = \sum_{n \neq m} \lim_{r \rightarrow 1^-} \frac{(1-r)c_n}{1 - re^{i\theta_m} e^{-i\theta_n}} = 0$$

Thus $\lim_{r \rightarrow 1^-} (1-r)f(re^{i\theta_m}) = c_m$, as asserted. □

The extended exterior of the open unit disk will be denoted by $\mathbb{D}_e = \{z \in \mathbb{C}_\infty : 1 < |z| \leq \infty\}$.

Theorem 8. *If $f(z) := \sum_{n=1}^{\infty} \frac{c_n}{1 - ze^{-i\theta_n}}$, $z \in \mathbb{D}$ and $f|_{\mathbb{D}} \equiv 0$, then $f|_{\mathbb{D}_e} \equiv 0$.*

Proof. If f is defined as in the previous theorem, and is identically 0 on \mathbb{D} , then all of the coefficients are 0. Thus, f is identically 0 everywhere where it is defined. \square

This is a type of coherence, and it is not the strongest result that we will get here. We will get a stronger form between the component functions $f|_{\mathbb{D}}$ and $f|_{\mathbb{D}_e}$ by matching nontangential boundary values. We will begin by first proving that they exist for the function f from Theorem 7 and Theorem 8.

Theorem 9. *Let $F \in \mathfrak{H}(\mathbb{D})$, where $\operatorname{Re}(F) > 0$. Then for all $0 < r < 1$ and $0 < p < 1$,*

$$\int_0^{2\pi} |F(re^{i\theta})|^p d\theta \leq A_p |F(0)|^p.$$

Here the constant $A_p \in (0, \infty)$ depends on p , but is independent of F and r .

Proof. Since $\operatorname{Re} F > 0$, $F = |F|e^{i\phi}$, where $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$. Furthermore, since F has no zeros in the disk by assumption, F^p is also analytic on \mathbb{D} and,

$$F^p = |F|^p (\cos(p\phi) + i \sin(p\phi)).$$

If $0 < p < 1$, noting that $\cos(p\phi)$ is minimized at $\phi = \frac{\pi}{2}$, and thus,

$$\operatorname{Re}(F^p) = |F|^p \cos(p\phi) \geq |F|^p \cos\left(\frac{p\pi}{2}\right).$$

Let $A_p = \sec(\frac{p\pi}{2})$. Then by the mean value property of harmonic functions, we have

$$\int_0^{2\pi} |F(re^{i\theta})|^p d\theta \leq A_p \int_0^{2\pi} \operatorname{Re}(F^p(re^{i\theta})) d\theta = A_p \operatorname{Re}(F^p(0)). \quad \square$$

Theorem 10. *If μ is a bounded complex Borel measure on \mathbb{T} , then*

$$f_\mu(z) := \int_{\mathbb{T}} \frac{1}{1 - e^{-it}z} d\mu(e^{it}) \in \bigcap_{0 < p < 1} H^p, \text{ (cf. Definition 4).}$$

In fact, $\|f_\mu\|_p^p \leq A_p \|\mu\|^p$, where $\|\mu\|$ is the total variation norm of μ and A_p is the constant from Theorem 9.

Proof. Consider the integrand of

$$f_\mu(z) := \int_{\mathbb{T}} \frac{1}{1 - e^{-it}z} d\mu(e^{it}),$$

and express it in the following form,

$$\begin{aligned} \frac{1}{1 - e^{-it}z} &= \frac{1}{1 - e^{-it}z} \left(\frac{1 - e^{it}\bar{z}}{1 - e^{it}\bar{z}} \right) \\ &= \frac{1 - e^{it}\bar{z}}{|1 - e^{-it}z|^2}. \end{aligned}$$

We may assume that μ is a real positive measure, since a general complex measure can be written as a linear combination of such measures. Also, noting that $\operatorname{Re}(z) = \operatorname{Re}(\bar{z})$, we obtain

$$\operatorname{Re}(f_\mu(z)) = \int_{\mathbb{T}} \frac{1 - \operatorname{Re}(e^{-it}z)}{|1 - e^{-it}z|^2} d\mu(e^{it}) \geq 0, \quad z \in \mathbb{D}.$$

Now we just apply Theorem 9 and note that $\|\mu\|^p \geq |f_\mu(0)|^p$ since holomorphic functions achieve their maximum on the boundary of their domain. \square

The function $f_\mu(z)$ from Theorem 7 is the “Cauchy transform” of the finite measure,

$$\mu = \sum_{n=1}^{\infty} c_n \delta_{e^{i\theta_n}}.$$

Thus, $f|_{\mathbb{D}}$ and $f|_{\mathbb{D}_e}$ both belong to H^p (resp. $H^p(\mathbb{D}_e) = \{F(z) : F(z) = f(1/z), \text{ where } f \in H^p\}$). Furthermore, the nontangential limits $f^*(z)$ exist a.e. on \mathbb{T} and $f^*(z) \in L^p(\mathbb{T})$ [15, Theorem 17.11(p.340)]. We must now check that these boundary values are equal for almost all $e^{i\theta}$. First note that for any bounded complex Borel measure μ on \mathbb{T} , the limits of $f_\mu(re^{i\theta})$ and $f_\mu(\frac{e^{i\theta}}{r})$ both exist as $r \rightarrow 1^-$. We will show that their nontangential limits are equal for almost every $e^{i\theta}$, when μ is singular with

respect to Lebesgue measure on the circle. To begin with,

$$\begin{aligned}
f_\mu(re^{i\theta}) - f_\mu\left(\frac{e^{i\theta}}{r}\right) &= \int_{\mathbb{T}} \frac{1}{1 - e^{-it}re^{i\theta}} d\mu(e^{it}) - \int_{\mathbb{T}} \frac{1}{1 - e^{-it}\frac{e^{i\theta}}{r}} d\mu(e^{it}) \\
&= \int_{\mathbb{T}} \frac{1 - \frac{1}{r}e^{-it}e^{i\theta} - 1 + re^{-it}e^{i\theta}}{1 - \frac{1}{r}e^{-it}e^{i\theta} - re^{-it}e^{i\theta} + (e^{-it}e^{i\theta})^2} d\mu(e^{it}) \\
&= \int_{\mathbb{T}} \frac{-e^{-it}e^{i\theta}(1 - r^2)}{r - e^{-it}e^{i\theta} - r^2e^{-it}e^{i\theta} + r(e^{-it}e^{i\theta})^2} d\mu(e^{it}) \\
&= \int_{\mathbb{T}} \frac{1 - r^2}{-re^{it}e^{i\theta} + 1 + r^2 - re^{-it}e^{i\theta}} d\mu(e^{it}). \\
&= \int_{\mathbb{T}} \frac{1 - r^2}{|e^{it} - re^{i\theta}|^2} d\mu(e^{it}).
\end{aligned}$$

This leaves us with the Poisson integral of μ . Consider the Lebesgue decomposition of the purely atomic measure μ , with respect to the normalized Lebesgue measure, σ , which is the arc length measure on \mathbb{T} . Let $E \subseteq \mathbb{T}$. By the theorem of Lebesgue-Radon-Nikodym (see Theorem 1) we can write uniquely,

$$\mu = \mu_a + \mu_s.$$

Moreover,

$$\mu = \sum_{n=1}^{\infty} c_n \delta_{e^{i\theta_n}}, \quad \delta_{e^{i\theta_n}} = \begin{cases} 1, & \text{if } e^{i\theta_n} \in E \\ 0, & \text{if } e^{i\theta_n} \notin E. \end{cases}$$

Let $A = \bigcup_n \{c_n\}$ and $B = \mathbb{T} \setminus A$. We now claim that $\mu \perp \sigma$. This can be seen by the following argument. Let

$$A \cap B = \emptyset,$$

$$\mu(E) = \mu(E \cap A), \quad \text{since } \mu \text{ is only dependent on } c_n$$

$\sigma(E) = \sigma(E \cap B)$ since we just remove a set of measure 0, which is $\sigma(A \cap E) = 0$.

If we let $\mu_a = 0$ for every set E , then $\sigma(E) = 0$ implies $\mu_a(E) = 0$ trivially. Therefore we will have $\mu \ll \sigma$. Now let $M \in L^1(\mathbb{T})$ denote the Radon-Nikodym derivative of its absolutely continuous part with respect to σ . Hence by Theorem 1,

$$\mu_a(E) = \int_E M d\sigma.$$

This means that $M = 0$, and by Fatou's Theorem (cf. Theorem 3),

$$\int \frac{1-r^2}{|e^{it}-re^{i\theta}|^2} d\mu(e^{it}) \rightarrow 0$$

for almost every $e^{i\theta}$ as $r \rightarrow 1^-$.

3.3. Generalization of L in Poincaré's example. Let L be the boundary of the union of two disjoint disks, $\{z_n\}$ be a sequence of distinct points in L which are dense on L , and $\{c_n\}$ be an absolutely summable sequence of nonzero complex numbers. If f is the function

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - z_n},$$

then we can use a similar technique as in Poincaré's example to match nontangential limits, obtaining a GAC across L from one disk to the other.

We begin by letting $\{\zeta_n\} = \{z_i\} \cup \{w_j\}$, where $\{z_i\} \subseteq A$, $\{w_j\} \subseteq B$, and the sets A and B are defined as follows:

$$L = \partial(A \cup B), \quad A = D(a, r_a) \quad \text{and} \quad B = D(b, r_b), \quad \text{where } A \cap B = \emptyset.$$

If we start in the interior of $D(a, r_a)$, by the same argument as in Poincaré's example, we will not have an analytic continuation across any point of $D(a, r_a)$. However, we do know that we have a GAC across $D(a, r_a)$ to $\mathbb{C} \setminus \overline{B}$ by the same arguments in the Poincaré example. These same arguments will work if we start with f in the interior of $D(b, r_b)$, so we have a GAC across $B(b, r_b)$ to $\mathbb{C} \setminus \overline{A}$. But our function $f(z)$ is holomorphic on the interior as well as on the exterior of disks A and B , so our continuations agree everywhere except possibly on $\{\zeta_n\}$. We also note that nontangential limits are unique, and thus we maintain a matching on A and B . So given a function of form

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - \zeta_n},$$

we can start in one disk, and find a GAC into another disjoint disk. This will generalize to a countable number of disjoint disks.

We now consider f and L defined in a similar way as the above generalization of the union of two disjoint disks, but L is now the boundary of two circles which share only one point. In this particular situation, we will use the same idea as before. If the point which the circles share happens to be one of the points in $\{z_n\}$, we just choose one set, A or B to assign it to. Now we will have a GAC from A to B in the same way as before. If the point which is shared is not one of the points of $\{z_n\}$, then there is nothing to prove and the GAC will still exist and hold.

A natural questions that will follow from these two generalizations is, “Is there a restriction to the shape of L ? Is there something special about circles?” It turns out that the condition of L being the union of two circles sharing just one point, or being disjoint from one another and having a GAC from one circle to the other, can be relaxed to simply connected smooth curves as well as regular n -gons.

4. CONTINUATION WITH ALMOST PERIODIC FUNCTIONS

So far we have looked at classical analytic continuation and a GAC. For the GAC, we have studied Poincaré’s example.

Let L be a smooth closed curve which bounds a convex set in the plane. Let $\{z_n\}$ be a sequence of distinct points dense in L , $\{c_n\}$ be an absolutely summable sequence of nonzero complex numbers, and let f be the function

$$f(z) := \sum_{n=1}^{\infty} \frac{c_n}{z - z_n}, \quad z \notin L.$$

We have been able to find a power series representation for the function f from Poincaré’s example; that is,

$$f(z) = \sum_{q=0}^{\infty} z^q B_q.$$

We will see that in the above displayed equation, the coefficient B_q , will suggest another type of coherence for $f|\mathbb{D}$, with $f|\mathbb{D}_e$, where \mathbb{D}_e is the exterior of the disk. This coherence property will be explored by looking at the function $q \mapsto B_q$, which

we will see is an almost periodic function on \mathbb{Z} , and by replacing $\{B_q\}$ by an almost periodic sequence.

4.1. Almost periodic functions: background information. Let us begin once again with some background information about almost periodic functions.

Definition 6. Let $f \in BUC(\mathbb{R})$ (Complex-valued, bounded, uniformly continuous functions on \mathbb{R} , endowed with the sup-norm). We will use the following notation $f_y(x)$ for the translations of f ,

$$f_y(x) = f(x - y), \quad y \in \mathbb{R},$$

and for $\lambda \in \mathbb{R}$, define

$$e_\lambda(x) = e^{i\lambda x}, \quad x \in \mathbb{R}.$$

Proposition 1. For a function $f \in BUC(\mathbb{R})$ the following two conditions are equivalent:

- (1) $\{f_y : y \in \mathbb{R}\}^-$ is compact in $BUC(\mathbb{R})$.
- (2) f is the sup-norm limit of finite linear combinations of the functions

$$\{e_\lambda : \lambda \in \mathbb{R}\}.$$

A function $f \in BUC(\mathbb{R})$ satisfying either of the equivalent conditions of the above proposition is said to be Bohr “almost periodic” (after H. Bohr, who first studied them in 1947). We will denote this class by $AP(\mathbb{R})$. It is also important to note the following definition, which was given by Bohr for almost periodic functions.

Definition 7. ([4, Property B (p. 14)]) A function $f \in BUC(\mathbb{R})$ is almost periodic if for any $\epsilon > 0$, there exists a number $l(\epsilon) > 0$ with the property that any interval of length $l(\epsilon)$ of the real line contains at least one point with abscissa ξ , such that

$$|f(x + \xi) - f(x)| < \epsilon, \quad -\infty < x < +\infty.$$

The number ξ is called the *translation number* of $f(x)$ corresponding to ϵ , or an ϵ -translation number.

Theorem 11. (Bohr [1])

For each $f \in AP(\mathbb{R})$, the following limit

$$M(f) := \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x) dx$$

exists.

Definition 8. The limit $M(f)$ from Theorem 11 is called the “Bohr mean value of f .”

Note that $M(f_y) = M(f)$, $M(f) \geq 0$ whenever $f \geq 0$, and $M(1) = 1$.

Definition 9. For $f \in AP(\mathbb{R})$ and $\lambda \in \mathbb{R}$, note that $e_{-\lambda}f \in AP(\mathbb{R})$. The “Bohr spectrum of f ” is defined to be the set

$$\Omega(f) := \{\lambda \in \mathbb{R} : M(e_{-\lambda}f) \neq 0\}.$$

Theorem 12. (Bohr [1])

If f is a non-trivial ($\neq 0$) almost periodic function, then $\Omega(f)$ is non-empty and moreover, $\Omega(f)$ is at most a countable (but not necessarily closed) set of real numbers.

Example 4. If

$$f(t) = \sum_{k=1}^K a_k e^{i\lambda_k t}, \quad \lambda_k \in \mathbb{R}, \quad a_k \in \mathbb{C}$$

then

$$M(e_{-\lambda}f) = \begin{cases} 0, & \text{if } \lambda \neq \lambda_k, k=1, 2, \dots, K, \\ a_k, & \text{if } \lambda = \lambda_k. \end{cases}$$

and thus

$$\Omega(f) = \{\lambda_1, \dots, \lambda_K\}.$$

The following observation is useful. If $f \in AP(\mathbb{R})$ and $\Omega(f) = \{\lambda_k\}$, then we can think of f as having a “Fourier” expansion

$$f \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k x},$$

where $a_k = M(e_{-\lambda_k}f)$. Moreover, we also have a “Parseval’s equality” ([4], p. 28)

$$\sum_{k=1}^{\infty} |a_k|^2 = M(|f|^2).$$

Theorem 13. ([4, Theorem 1.19 (p, 29)])

Two distinct $AP(\mathbb{R})$ functions have distinct Fourier expansions.

Proof. If $f, g \in AP(\mathbb{R})$ and $f(x) \neq g(x)$ had the same Fourier series, then from Parseval's equality applied $f(x) - g(x)$ would yield

$$M(|f(x) - g(x)|^2) = \sum_{k=1}^{\infty} |c_k|^2 = 0,$$

where $c_k = 0$ is the difference between coefficients of f and g . Therefore, it is sufficient to show that a nonnegative and non-vanishing almost periodic function has positive "Bohr mean." Let $\phi(x) \in AP(\mathbb{R})$ and $\phi(x) \geq 0$ and $\phi(x_0) = \alpha > 0$. Choose two numbers $l > 0$ and $\delta > 0$ such that any interval of length l will contain an interval of length 2δ . Furthermore, the points of the interval 2δ must all be $(\alpha/3)$ -translation numbers of $\phi(x)$, and $|x_1 - x_2| < \delta$ should imply $|\phi(x_1) - \phi(x_2)| < \alpha/3$. Consider any interval of length l ; i.e., $(a - \delta - x_0, a + l - \delta - x_0)$, where a is a real number. Then there exists an $(\alpha/3)$ -translation number β of $\phi(x)$ which belongs to this interval. We can see that $x_0 + \xi \in (a - \delta, a + l - \delta)$, and assuming that $|x - x_0| < \delta$, the number $x + \xi$ will range over an interval of length 2δ . Note the following,

$$\phi(x + \xi) = \phi(x_0) + [\phi(x) - \phi(x_0)] + [\phi(x + \xi) - \phi(x)] > \alpha - \frac{\alpha}{3} - \frac{\alpha}{3} = \frac{\alpha}{3}.$$

This shows that any interval of length l on the real line contains a subinterval of length 2δ with $\phi(x) > \alpha/3$ at all points in this subinterval. Furthermore, this implies

$$\frac{1}{nl} \int_0^n l\phi(x)dx = \frac{1}{nl} \sum_{k=1}^n \int_{(k-1)l}^k l\phi(x)dx > \frac{1}{nl} n(2\delta)\left(\frac{\alpha}{3}\right) = \frac{2\alpha\delta}{3l}.$$

Now by letting $n \rightarrow \infty$, we obtain

$$M(\phi(x)) \geq \frac{2\alpha\delta}{3l} > 0.$$

□

Also, if $f \in AP(\mathbb{R})$ then by Proposition 1(2) there exist trigonometric polynomials

$$\sigma_m(x) = \sum_{k=1}^{n_m} r_{k,m} a_k e^{i\lambda_k x}$$

which converge uniformly to f as $m \rightarrow \infty$ on \mathbb{R} . The $r_{k,m}$ are rational and depend on λ_k and m , but not on a_k . Therefore, almost periodic functions can be approximated by a sequence of trigonometric polynomials satisfying $\Omega(\sigma_m) \subseteq \Omega(f)$.

Definition 10. *The class of sequences in $\ell^\infty(\mathbb{Z})$, where we denote the set of bounded two-sided sequences of complex numbers with the norm*

$$\|A\| = \sup\{|A(n)| : n \in \mathbb{Z}\},$$

satisfying one of the following equivalent conditions in Proposition 2 below, is called the class of “almost periodic sequences” and is denoted by $AP(\mathbb{Z})$.

Proposition 2. *For $A \in \ell^\infty(\mathbb{Z})$, the following are equivalent:*

- (1) *The set $\{A(\cdot - m) : m \in \mathbb{Z}\}^-$ is compact in $\ell^\infty(\mathbb{Z})$.*
- (2) *A is the norm limit of finite linear combinations of the characters*

$$\{n \rightarrow e^{in\lambda} : \lambda \in [0, 2\pi)\}.$$

- (3) *There exists an $f \in AP(\mathbb{R})$ such that $f(n) = A(n)$ for all $n \in \mathbb{Z}$.*

Recall the “Bohr mean value” of $A \in AP(\mathbb{R})$, (cf. Definition 8) and we now similarly define the “Bohr mean value” of $A \in AP(\mathbb{Z})$ as follows:

$$M(A) := \lim_{m \rightarrow \infty} \frac{1}{2m+1} \sum_{k=-m}^m A(k)$$

and the “Bohr spectrum” of A as

$$\Omega(A) := \{e^{i\theta} : M_\theta(A) := \lim_{m \rightarrow \infty} \frac{1}{2m+1} \sum_{k=-m}^m A(k)e^{-ik\theta} \neq 0\}.$$

Proposition 3. *Let $f \in AP(\mathbb{R})$. If $f(n) = A(n)$ for all $n \in \mathbb{Z}$, then $M_\theta(A) = M(e_{-\theta}f)$ for all θ . Moreover, $\Omega(f) = \Omega(A)$.*

Proof. Let $\epsilon > 0$. Since $f \in AP(\mathbb{R})$, f can be approximated by a sequence of trigonometric polynomials, $\sigma_m(x) = \sum_{k=1}^{n_m} r_{k,m} a_k e^{i\lambda_k x}$, where $|f - \sigma_m(x)| < \epsilon$.

Thus,

$$\begin{aligned}
|M(e_{-\theta}f) - M(e_{-\theta}\sigma_m(x))| &= \lim_{T \rightarrow \infty} \frac{1}{2T} \left| \int_{-T}^T e^{-i\theta x} (f(x) - \sigma_m(x)) dx \right| \\
&\leq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |e^{-i\theta x} (f(x) - \sigma_m(x))| dx \\
&\leq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |e^{-i\theta x}| |\epsilon| dx \\
&= \epsilon.
\end{aligned}$$

We may now show that $M_\theta(A) = M(e_{-\theta}\sigma_m(x))$ for all θ . By the linearity of the limit and integral operators, it will suffice to show that $M_\theta(A) = M(e_{-\theta}f)$ when $f(x) = ae^{i\theta_n x}$. For $M(e_{-\theta}ae^{i\theta_n x})$ we have,

$$M(e_{-\theta}ae^{i\theta_n x}) = \lim_{T \rightarrow \infty} \int_{-T}^T ae^{i\theta_n x} e^{i\theta x} dx.$$

Thus,

$$M(e_{-\theta}ae^{i\theta_n x}) = \begin{cases} 0, & \text{if } \theta \neq \theta_n \\ a, & \text{if } \theta = \theta_n \end{cases},$$

and similarly,

$$M_\theta(A) = \lim_{m \rightarrow \infty} \frac{1}{2m+1} \sum_{k=-m}^m e^{i\theta_n k} e^{ik\theta}.$$

Therefore,

$$M_\theta(A) = \begin{cases} 0, & \text{if } \theta \neq \theta_n \\ a, & \text{if } \theta = \theta_n \end{cases}.$$

The case when $\theta_n \neq \theta$ is a little “tricky” to compute for $M_\theta(A)$, so it is provided here;

$$\begin{aligned}
M_\theta(A) &= \lim_{m \rightarrow \infty} \frac{1}{2m+1} \sum_{k=-m}^m e^{i\theta_n k} e^{ik\theta} \\
&= \lim_{m \rightarrow \infty} \frac{1}{2m+1} \sum_{k=-m}^m e^{i(\theta_n - \theta)k} \\
&= \lim_{m \rightarrow \infty} \frac{1}{2m+1} \frac{1 - e^{i(\theta_n - \theta)(m+1)}}{1 - e^{i(\theta_n - \theta)}} = 0
\end{aligned}$$

□

Proposition 4. *If $A \in AP(\mathbb{Z})$, then A can be approximated, in the norm of $\ell^\infty(\mathbb{Z})$, by a sequence $\{A_n\}$ which is a finite linear combination of characters where $\Omega(A_n) \subset \Omega(A)$.*

Proof. If $A \in AP(\mathbb{Z})$, then there exists $f \in AP(\mathbb{R})$ such that $f(n) = A(n)$ for all $n \in \mathbb{N}$. By Proposition 3, $\Omega(f) = \Omega(A)$. As remarked earlier from Proposition 1(2), there exists a trigonometric polynomial $\sigma_n \rightarrow f$ uniformly such that $\Omega(\sigma_n) \subset \Omega(f)$. So the required sequence $\{A_n\}$ is given by $A_n = \{\sigma_n(k)\}$. □

Proposition 5. *If $A \in AP\{\mathbb{Z}\}$ and $A(n) = 0$ for all $n \geq 0$, then $A(n) = 0$ for all $n \in \mathbb{Z}$.*

Proof. We proceed by contradiction by assuming $A(-1) \neq 0$. Then for $K, L \in \mathbb{N} \cup \{0\}$ and $K \neq L$,

$$\sup\{|A(n-K) - A(n-L)| : n \in \mathbb{Z}\} \geq |A(-1)| \neq 0.$$

Thus the sequence

$$\{\{A(\cdot - K)\} : K \geq 0\}$$

doesn't have a convergent subsequence in $\ell^\infty(\mathbb{Z})$. But this contradicts the compactness condition for $A \in \ell^\infty(\mathbb{Z})$. Continuing by induction on n in $A(n)$ for $n \leq -2$, will show that all $A(n) = 0$ for $n \in \mathbb{Z}$. □

4.2. Compatibility with Analytic Continuation.

Lemma 1. *Suppose $A \in AP(\mathbb{Z})$ and*

$$f_A(z) = \sum_{n=0}^{\infty} A(n)z^n, \quad z \in \mathbb{D}.$$

Then

$$M_{-\theta}(A) = \lim_{r \rightarrow 1^-} (1-r)f_A(re^{i\theta}) \text{ for all } \theta \in [0, 2\pi).$$

Proof. Let $\epsilon > 0$. Since $A \in AP(\mathbb{Z})$, we can find a $B(n)$ as follows,

$$B(n) = \sum_{k=1}^K b_k e^{in\theta_k}, \quad n \in \mathbb{Z},$$

where $\|A - B\| < \epsilon/2$. Also,

$$\begin{aligned}
 f_B(z) &= \sum_{n=0}^{\infty} \sum_{k=1}^K b_k e^{in\theta_k} z^n \\
 &= \sum_{n=0}^{\infty} \sum_{k=1}^K b_k (e^{i\theta_k} z)^n \\
 &= \sum_{k=1}^K b_k \sum_{n=0}^{\infty} (e^{i\theta_k} z)^n \\
 &= \sum_{k=1}^K \left[b_k \frac{1}{1 - e^{i\theta_k} z} \right] \\
 &= \sum_{k=1}^K \frac{b_k}{1 - e^{i\theta_k} z}.
 \end{aligned}$$

Moreover,

$$|M_{\theta}(A) - M_{\theta}(B)| \leq \|A - B\| < \epsilon/2 \text{ for all } \theta \in [0, 2\pi).$$

Now we use the “give and take” method to rewrite the following,

$$f_A(re^{i\theta}) = f_B(re^{i\theta}) + [f_A(re^{i\theta}) - f_B(re^{i\theta})].$$

We can also note the following inequality,

$$(1-r)|f_A(re^{i\theta}) - f_B(re^{i\theta})| \leq (1-r) \sum_{n=0}^{\infty} |A(n) - B(n)| r^n \leq \|A - B\| \leq \epsilon/2,$$

and with the our new representation for $f_B(z)$,

$$|(1-r)f_A(re^{i\theta}) - M_{-\theta}(A)| \leq |(1-r) \sum_{k=1}^K \frac{b_k}{1 - e^{i\theta_k} r e^{i\theta}} - M_{-\theta}(A)| + \epsilon/2.$$

From before we have,

$$\lim_{r \rightarrow 1^-} (1-r) \sum_{k=1}^K \frac{b_k}{1 - e^{i\theta_k} r e^{i\theta}} = M_{-\theta}(B),$$

and thus,

$$\limsup_{r \rightarrow 1^-} |(1-r)f_A(re^{i\theta}) - M_{-\theta}(A)| \leq |M_{-\theta}(B) - M_{-\theta}(A)| + \epsilon/2.$$

With our new inequality above, we now have that this is bounded above by ϵ . \square

The above lemma implies that

$$|f_A(re^{i\theta})| \rightarrow +\infty \text{ as } r \rightarrow 1^-$$

for each $e^{i\theta} \in \Omega(A)$.

Before proceeding, the following theorem is needed, which is from a paper of Sundberg [16].

Lemma 2. *Let U be an open set in \mathbb{C} and \mathcal{F} be a family of functions from $\mathfrak{H}(U)$ (holomorphic functions on U). If there is a $p \in L^1(U, dA)$ such that*

$$\log^+ |f(z)| \leq p(z), \quad \text{for all } f \in \mathcal{F} \text{ and } z \in U,$$

then \mathcal{F} is a normal family.

Proof. By Montel's theorem, it will suffice to show,

$$\sup\{|f(z)| : z \in K, f \in \mathcal{F}\} < \infty, \quad \text{for any } K_{\text{compact}} \subset U.$$

Let $K_{\text{compact}} \subset U$, and let $\delta > 0$ such that,

$$K_\delta := \{z \in \mathbb{C} : \text{dist}(z, K) \leq \delta\} \subset U.$$

If $f \in \mathcal{F}$ and $x \in K$, then the subharmonicity of $\log^+ |f|$ implies the following,

$$\log^+ |f(z)| \leq \frac{1}{\pi\delta^2} \int_{|w-z| \leq \delta} \log^+ |f(w)| dA(w) \leq \frac{1}{\pi\delta^2} \int_{K_\delta} p(w) dA(w).$$

Thus,

$$|f(z)| \leq \exp \left(\frac{1}{\pi\delta^2} \int_{K_\delta} p(w) dA(w) \right) \text{ for all } f \in \mathcal{F} \text{ and } z \in K.$$

□

Theorem 14. *Suppose $A \in AP(\mathbb{Z})$ and*

$$f_A(z) = \sum_{n=0}^{\infty} A(n)z^n, \quad z \in \mathbb{D},$$

$$F_A(z) = - \sum_{n=1}^{\infty} \frac{A(-n)}{z^n}, \quad z \in \mathbb{D}_e.$$

If f_A has an analytic continuation across some boundary arc $J \subset \mathbb{T}$, then this analytic continuation must be equal to F_A .

Proof. If f_A has an analytic continuation across some boundary arc $J \subset \mathbb{T}$, then $f_A(re^{i\theta})$ remains bounded as $r \rightarrow 1^-$ for each $e^{i\theta} \in J$. By Lemma 2 we just proved, this means that $M_{-\theta}(A) = 0$, thus $e^{i\theta} \notin \Omega(A)$ for all $e^{i\theta} \in J$. Thus, we can approximate A by a sequence $\{A_s : s = 1, 2, 3, \dots\}$ of $AP(\mathbb{Z})$ sequences of the form

$$A_s(n) = \sum_{k=1}^{K_s} a_{s,k} e^{in\theta_{s,k}}, n \in \mathbb{Z},$$

and

$$e^{i\theta_{s,k}} \notin J.$$

Using the uniform norm $\|\cdot\|$, we can arrange that $\|A - A_s\| \leq 1/s$. Also,

$$f_{A_s}(z) = \sum_{k=1}^{K_s} \frac{a_{s,k}}{1 - e^{i\theta_{s,k}} z} := R_s(z), \quad |z| < 1,$$

as before; and $F_{A_s}(z) = R_s(z)$, $|z| > 1$. Now note the following,

$$|f_A(z) - R_s(z)| \leq \sum_{n=0}^{\infty} |A(n) - A_s(n)| |z|^n \leq \frac{1}{s} \frac{1}{1 - |z|}, \quad |z| < 1,$$

$$|f_A(z) - R_s(z)| \leq \frac{1}{s} \frac{1}{|z| - 1}, \quad |z| > 1.$$

Let γ be a circle with its center in J , and does not contain either of the endpoints of J . From the above inequalities we get

$$|R_s(z)| \leq \frac{C}{||z| - 1|}, \quad z \in \text{int}(\gamma),$$

where C is some positive constant, which is independent of s . We now have a normal family on the interior of γ , which is the sequence $\{R_s\}$, and thus there is a subsequence which converges uniformly on compact subsets of the interior of γ . With the estimates above, we now have that F_A is an analytic continuation of f_A across J . \square

Now we will look at a version of the above theorem, but with F_A and f_A replaced by the following Laplace transforms,

$$f_\phi(z) = \int_0^\infty \phi(t)e^{-tz} dt, \quad z = x + iy, \quad x > 0,$$

$$F_\phi(z) = - \int_{-\infty}^0 \phi(t)e^{-tz} dt, \quad z = x + iy, \quad x < 0,$$

and $\phi \in AP(\mathbb{R})$. Just as in the discrete case, $f_\phi \equiv 0$ implies $\phi|_{\mathbb{R}_+} = 0$, which by almost periodicity yields $\phi|_{\mathbb{R}_-} = 0$, and thus, $F_\phi \equiv 0$. In this sense, they uniquely determine each other and the proof is the same as in the discrete case. The main goal from here is to show the compatibility of the continuation from $f_\phi \rightarrow F_\phi$ with analytic continuation.

Lemma 3. For $\phi \in AP(\mathbb{R})$,

$$\lim_{x \rightarrow 0^+} x f_\phi(x + iy) = M(e_{-y}\phi).$$

Proof. Let $\epsilon > 0$ and let

$$\phi_n(t) = \sum_{k=1}^K a_{n,k} e^{it\lambda_{n,k}}$$

be such that $|\phi_n(t) - \phi(t)| < \epsilon$ for all $t \in \mathbb{R}$. Note the following computation with $z = x + iy$ and $x > 0$,

$$\begin{aligned} \int_0^\infty e^{it\lambda_{n,k}} e^{-tz} dt &= \int_0^\infty e^{t(i\lambda_{n,k} - z)} dt \\ &= \lim_{m \rightarrow \infty} \left. \frac{e^{t(i\lambda_{n,k} - z)}}{i\lambda_{n,k} - z} \right|_0^m \\ &= \frac{1}{z - i\lambda_{n,k}}. \end{aligned}$$

Thus,

$$f_{\phi_n}(z) = \sum_{k=1}^K \frac{a_{n,k}}{z - i\lambda_{n,k}}$$

and so

$$f_\phi(z) = \sum_{k=1}^K \frac{a_{n,k}}{z - i\lambda_{n,k}} + \int_0^\infty [\phi(t) - \phi_n(t)] e^{-tz} dt.$$

We now have,

$$|x f_\phi(x + iy) - M(e_{-y}\phi)| \leq \left| \sum_{k=1}^K \frac{x a_{n,k}}{x + iy - i\lambda_{n,k}} - M(e_{-y}\phi) \right| + \epsilon.$$

By Example 4 we have,

$$\lim_{x \rightarrow 0^+} \sum_{k=1}^K \frac{x a_{n,k}}{x + iy - i \lambda_{n,k}} = M(e_{-y} \phi_n).$$

Thus,

$$|x f_\phi(x + iy) - M(e_{-y} \phi)| \leq \epsilon.$$

□

Theorem 15. *Suppose $\phi \in AP(\mathbb{R})$ and f_ϕ and F_ϕ are defined as above. If f_ϕ has an analytic continuation across some interval $(ia, ib) \subset i\mathbb{R}$, then this continuation must be equal to F_ϕ .*

Proof. Suppose f_ϕ is analytic across $(ia, ib) \subset i\mathbb{R}$. Since $f(x + iy)$ is bounded for $y \in (a, b)$, the lemma above tells us that $M(e_{-y} \phi) = 0$ for $y \in (a, b)$ and thus $\Omega(\phi) \cap (a, b) = \emptyset$. By Proposition 1(2) ϕ can be approximated uniformly by ϕ_n of the form,

$$\phi_n(t) = \sum_{k=1}^{K_n} a_{n,k} e^{it \lambda_{n,k}}, \quad \lambda_{n,k} \notin (a, b).$$

We can also arrange things such that $|\phi(t) - \phi_n(t)| \leq 1/n$ for all t . Thus,

$$f_{\phi_n}(z) = \sum_{k=1}^K \frac{a_{n,k}}{z - i \lambda_{n,k}} := R_n(z).$$

There is a similar expression for $F_\phi(z)$. Note the following,

$$|f_\phi(z) - R_n(z)| \leq \int_0^\infty |\phi(t) - \phi_n(t)| e^{-tz} dt \leq \frac{1}{n} \frac{1}{x}, \quad z = x + iy, \quad x > 0,$$

$$|f_\phi(z) - R_n(z)| \leq \frac{1}{n} \frac{1}{|x|}, \quad z = x + iy, \quad x < 0.$$

Let γ be a circle with its center in (ia, ib) , and does not contain either of the endpoints. From the above inequalities we get

$$|R_n(z)| \leq \frac{C}{||z| - 1|}, \quad z \in \text{int}(\gamma),$$

where C is some positive constant, which is independent of n . We now have a normal family on the interior of γ , which is the sequence $\{R_n\}$, and thus there is a subsequence which converges uniformly on compact subsets of the interior of γ . With the estimates above, we now have that F_A is an analytic continuation of f_A across (ia, ib) . \square

5. FUNCTIONS WITHOUT GENERALIZED CONTINUATIONS

5.1. Background and Definition. So far we have discussed a function which was analytic in \mathbb{D} and didn't have an analytic continuation across \mathbb{T} , but it did however have a generalized analytic continuation across \mathbb{T} to the exterior of the disk. We have also talked about continuations of almost periodic functions and their coherence property, which if they existed, they were uniquely determined by each other. Pseudocontinuation, as we will see, encompasses the Poincaré example, as well as the almost periodic functions that we have talked about in the class \mathfrak{U} . The natural question is whether all analytic functions have a pseudocontinuation across their natural boundaries. First we begin by defining a more generalized type of continuation.

Definition 11. (*Pseudocontinuation*) Let Ω be a region (an open connected set) in the extended exterior disk \mathbb{D}_e which shares a non-degenerate boundary arc I with the unit disk \mathbb{D} . We say that $T_f \in \mathfrak{M}(\Omega)$ (meromorphic on Ω) is a “pseudocontinuation” of $f \in \mathfrak{M}(\mathbb{D})$ across I if the nontangential limits of T_f and f exist and are equal almost everywhere on I .

Definition 12. Define

$$\mathfrak{N}(\mathbb{D}_e) := \left\{ \frac{G}{H} : G, H \in H^\infty(\mathbb{D}_e) \right\}$$

to be the “functions of bounded type” (in the Nevanlinna's sense) on \mathbb{D}_e . Also, set *PCBT* to be the class of $f \in H^2$ which have a pseudocontinuation across \mathbb{T} to a function $T_f \in \mathfrak{N}(\mathbb{D}_e)$.

It is important to make the following remarks about functions in $\mathfrak{N}(\mathbb{D}_e)$.

- (1) If $\lim_{z \rightarrow 0} G(1/z) = 0$ and $\lim_{z \rightarrow 0} H(1/z) \neq 0$, then $\lim_{z \rightarrow 0} \frac{G(1/z)}{H(1/z)} = 0$.
- (2) If $\lim_{z \rightarrow 0} G(1/z) \neq 0$ and $\lim_{z \rightarrow 0} H(1/z) \neq 0$, then $\lim_{z \rightarrow 0} \frac{G(1/z)}{H(1/z)} < \infty$.

(3) If $\lim_{z \rightarrow 0} G(1/z) \neq 0$ and $\lim_{z \rightarrow 0} H(1/z) = 0$, then $\lim_{z \rightarrow 0} \frac{G(1/z)}{H(1/z)} = \infty$, with a pole at 0.

(4) If $\lim_{z \rightarrow 0} G(1/z) = 0$ and $\lim_{z \rightarrow 0} H(1/z) = 0$, then

$$\lim_{z \rightarrow 0} \frac{G(1/z)}{H(1/z)} = \lim_{z \rightarrow 0} \frac{z^n g(z)}{z^m h(z)}, \quad \text{where } z^n g(z) = G(1/z) \text{ and } z^m h(z) = H(1/z).$$

If $m > n$, then $\lim_{z \rightarrow 0} \frac{z^n g(z)}{z^m h(z)} = \infty$, with a pole at 0. If $n > m$, then $\lim_{z \rightarrow 0} \frac{z^n g(z)}{z^m h(z)} = 0$.

The following remarks can be made about pseudocontinuations.

- (1) If $f \in \mathfrak{M}(\mathbb{D})$ has a pseudocontinuation $T_f \in \mathfrak{M}(\Omega)$ across I , it is unique. This comes from Lusin-Privalov's uniqueness theorem.
- (2) The "coherence" property of pseudocontinuation with analytic continuation is that, if $f \in \mathfrak{M}(\mathbb{D})$ has a pseudocontinuation $T_f \in \mathfrak{M}(\Omega)$ across I , and f has an analytic continuation to a neighborhood U of a boundary point, then $T_f = f$ on $U \cap \Omega$.

5.2. Gap Series. In complex analysis, gap series, which are Taylor series of the form

$$\sum_{n=0}^{\infty} a_n z^{\lambda_n}, \quad \lambda_0 < \lambda_1 < \lambda_2 < \dots$$

where the λ_n 's are a scarce subset of the integers, play an important role in the understanding of analytic continuation of Taylor series across their circles of convergence. In 1872, Weierstrass gave his famous example of a Fourier gap series

$$\sum_{n=1}^{\infty} a^n \cos(\lambda^n \theta),$$

(with $\lambda \geq 3$ is an odd integer, $0 < a < 1$ and $a\lambda > 1 + 3\pi/2$) which is continuous but nowhere differentiable with the real variable θ . Hardy [11] was able to relax the condition of Weierstrass's example, thus instead of $0 < a < 1$ and $a\lambda > 1 + 3\pi/2$, we get $0 < a < 1$ and $a\lambda > 1$ as one of the conditions for Weierstrass's example. The non-differentiability on the boundary of the unit circle is not the only reason that a Taylor series can have the unit circle as a natural boundary. This was pointed out by Fredholm [7], [8] in 1890 which was shown with the use of the heat equation. His argument originally contained an error, but was corrected in [12].

A large class of functions with the unit circle as a natural boundary was found in 1892 by Hadamard, and his theorem is below.

Theorem 16. (*Hadamard [10]*)

A Taylor series of the form,

$$\sum_{n=0}^{\infty} a_n z^{\lambda_n}, \quad \lambda_0 < \lambda_1 < \lambda_2 < \dots,$$

with radius of convergence equal to one, and in addition satisfying

$$\frac{\lambda_{n+1}}{\lambda_n} \geq q > 1, n = 0, 1, 2, \dots$$

has the unit circle as a natural boundary.

In 1898, Fabry was able to improve Hadamard's gap theorem and further expand Hadamard's class of functions.

Theorem 17. (*Fabry [6]*)

A Taylor series of the form,

$$\sum_{n=0}^{\infty} a_n z^{\lambda_n}, \quad \lambda_0 < \lambda_1 < \lambda_2 < \dots,$$

and with radius of convergence equal to one, which in addition satisfies

$$\lim_{n \rightarrow \infty} \frac{n}{\lambda_n} = 0,$$

has the unit circle as a natural boundary.

By virtue of Pólya's following theorem (see also Erdős [5]), the condition of the limit from Theorem 17 is about as relaxed as can be.

Theorem 18. (*Pólya [13]*)

Suppose, for fixed integers $\lambda_0 < \lambda_1 < \lambda_2 < \dots$, every series with radius of convergence one, of the form

$$\sum_{n=0}^{\infty} a_n z^{\lambda_n}, \quad \lambda_0 < \lambda_1 < \lambda_2 < \dots,$$

has the unit circle as a natural boundary. Then

$$\lim_{n \rightarrow \infty} \frac{n}{\lambda_n} = 0.$$

Theorem 19. *If*

$$f(z) = \sum_{n=0}^{\infty} 2^{-n} z^{2^n},$$

then f (Note that f is continuous on \mathbb{D}^- and has a radius of convergence equal to one) does not have a pseudocontinuation across any arc of \mathbb{T} .

For the purpose of this Master's paper, the proof that $f(z) \notin PCBT$ is provided below. A full proof of this theorem can be found on [14, p. 78].

Proof. If $f \in PCBT$, then there is a $T_f \in \mathfrak{N}(\mathbb{D}_e)$ which is a pseudocontinuation of f across \mathbb{T} . If λ_k is a primitive $2^k - th$ root of unity, then

$$\begin{aligned} f(z) - f(\lambda_k z) &= \sum_{n=0}^{\infty} 2^{-n} z^{2^n} - \sum_{n=0}^{\infty} 2^{-n} (\lambda_k z)^{2^n} \\ &= \sum_{n=0}^k 2^{-n} z^{2^n} - \sum_{n=0}^k 2^{-n} (\lambda_k z)^{2^n} + \left(\sum_{n>k}^{\infty} 2^{-n} z^{2^n} - \sum_{n>k}^{\infty} 2^{-n} (\lambda_k z)^{2^n} \right) \\ &= \sum_{n=0}^k 2^{-n} z^{2^n} - \sum_{n=0}^k 2^{-n} (\lambda_k z)^{2^n} + \left(\sum_{n>k}^{\infty} [2^{-n} z^{2^n} - 2^{-n} (\lambda_k z)^{2^n}] \right) \\ &= \sum_{n=0}^k 2^{-n} z^{2^n} - \sum_{n=0}^k 2^{-n} (\lambda_k z)^{2^n} \\ &= p_k(z), \quad z \in \mathbb{D}, \end{aligned}$$

for some polynomial p_k . Since the above equation holds almost everywhere on \mathbb{T} , then

$$T_f(z) - T_f(\lambda_k z) = p_k(z), \quad z \in \mathbb{D}_e,$$

since we are assuming that T_f is the pseudocontinuation of $f(z)$. For a suitable choice of k , the degree of p_k can be made as large as desired, thus T_k will not have at worst a pole at ∞ . This contradicts that $T_f \in \mathfrak{N}(\mathbb{D}_e)$, and finishes the proof. \square

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